ISSN: 2456-1657

Elins International Journal

Elins International Journal of Science Engineering & Management (EIJSEM)
Special Issue on Application of Mathematics in Physics and Engineering, Volume 4, Issue 4, May 2020

Double Dirichlet Average of Modified Bessel Function Via Fractional Derivative

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Abstract: In this paper ,the result of double Dirichlet average of Modified Bessel function using Reimann-Liouville fractional derivative then a relation between Modified bessel function and fractional derivative is established .

Keywords and Phrases: Dirichlet average, Modified Bessel function fractional derivative and Fractional calculus operators.

Mathematics Subject Classification: 26A33, 33A30, 33A25 and 83C99.

1. Introduction

Carlson [2-5] has a credit to define first time Dirichlet average of functions which shows special type of integration average with respect to Dirichlet measure. He represented that various important special functions can be obtained as Dirichlet averages for the elementary functions like *polynomial and exponential function* etc. He has also found [3] that the hidden symmetry of all special functions which provided their various transformations can be obtained by averaging *polynomial and exponential function* x^n, e^x etc. Thus he made first time a process towards the unification of special functions by averaging a limited number of ordinary functions [6,7].More or less all recognized special functions and their well known properties have been derived by this unique process.

Gupta and Agarwal [11,12] define that averaging process is depend upon the old theory of fractional derivative use the old theory the averaging process is converted into new theory. In his work we can see in so many cases Carlson applied fractional derivative. Deora and Banerji have introduce of double Dirichlet Average [7] and triple Dirichlet Average[8] of e^x *and* x^t respectively by the using of fractional derivative .

Double Dirichlet average and triple Dirichlet average are also defined by Sharma and Jain [26] .They have found the double Dirichlet average of trigonometric function cosx by using fractional derivative and triple Dirichlet average of e^x by using fractional calculus

Kilbas and Kattuveetti [13] defined a relation between Dirichlet average of the generalized Mittag-Leffler function with Riemann –Liouville fractional integrals and the hyper-geometric function of many variables.

The Dirichlet averages of Generalized Multi- index Mittag –Leffler function is introduced by Saxena, pongany, Ram and Daiya [5] in terms of Riemann –Liouville fractional integral and hypergeometric functions of several variables functions

In this chapter, we have established the relations among single, double and triple Dirichlet averages Matheiu –type series, using fractional derivative.

2. Definitions

We give blew some of the definitions which are necessary in the preparation of this paper.

2.1 Standard Simplex in $R^n, n \geq 1$:

We denote the standard simplex in R^n, $n \geq 1$ by [1, p.62].

$$E = E_n = \{S(u_1, u_2, \dots u_n) : u_1 \geq 0, \dots u_n \geq 0,\ u_1 + u_2 \dots + u_n \leq 1\} \qquad (2.1.1)$$

2.2 Dirichlet measure:

Let $b \in C^k, k \geq 2$ and let $E = E_{k-1}$ be the standard simplex in R^{k-1}. The complex measure μ_b is defined by $E[1]$.

$$d\mu_b(u) = \frac{1}{B(b)} u_1^{b_1-1} \dots u_{k-1}^{b_{k-1}-1}(1-u_1-\cdots - u_{k-1})b_k^{-1} du_1 \dots du_{k-1} \quad (2.2.1)$$

Will be called a Dirichlet measure.

Here

$$B(b) = B(b1, \dots bk) = \frac{\Gamma(b_1)\dots\Gamma(b_k)}{\Gamma(b_1+\cdots+b_k)},$$
$$C_> = \{z \in z: z \neq 0, |ph\, z| < \pi/2\},$$

Open right half plane and $C_>$k is the k^{th} Cartesian power of $C_>$

2.3 Dirichlet Average[1, p.75]:

Let Ω be the convex set in $C_>$, let $z = (z_1, \dots\dots\dots, z_k) \in \Omega^k, k \geq 2$ and let $u.z$ be a convex combination of $z_1, \dots\dots\dots, z_k$. Let f be a measureable function on Ω and let μ_b be a Dirichlet measure on the standard simplex E in R^{k-1}. Define

$$F(b,z) = \int_E f(u.z) d\,\mu_b(u) \quad (2.3.1)$$

We shall call F the Dirichlet measure of f with variables

$z = (z_1, \dots\dots\dots, z_k)$ and parameters $b = (b_1, \dots\dots\dots. b_k)$.

Here

$$u.z = \sum_{i=1}^{k} u_i z_i \text{ and } u_k = 1 - u_1 - \cdots \dots\dots. - u_{k-1} \quad (2.3.2)$$

If $k = 1$, define $F(b,z) = f(z)$.

2.4 Fractional Derivative [9, p.181]:

The concept of fractional derivative with respect to any function has been used by Erdelyi[8]. The common definition for the fractional derivative of order α found in the literature on the "Riemann-Liouville integral" is

$$D_z^{\alpha} F(z) = \frac{1}{\Gamma(-\alpha)} \int_0^z F(t)(z-t)^{-\alpha-1} dt \quad (2.4.1)$$

Where $Re(\alpha) < 0$ and $F(x)$ is the form of $x^p f(x)$, where $f(x)$ is analytic at $x = 0$

2.5 Definitions and Preliminaries:

Double Averages of Functions of One Variable (From [2,3]):

let z be a $k \times x$ matrix with complex elements z_{ij}. Let $u = (u_1, \dots u_k)$ be an ordered of k-tuple of real non negative weight $\sum u_i = 1$ and $v = (v_1, \dots v_k)$ be an ordered of x-tuple of real non-negative weight $\sum v_i = 1$

$$u.z.v = \sum_{i=1}^{k} \sum_{j=1}^{x} u_i z_{ij} v_j \quad (2.5.1)$$

If z_{ij} is regarded as a point of the complex plane, all these convex combinations are points in the convex hull of $(z_{11}, \dots z_{kx})$, denote by $H(z)$.

Let $b = (b_1, \dots b_k)$ be an ordered k –tuple of complex numbers with positive real part (Re (b) > 0) and similarly for β=$(\beta_1, \dots \beta_x)$. Then we define $d\mu_b(u)$ and $d\mu_b(v)$.

Let f be the holomorphic on a domain D in the complex plane, If $Re(b) > 0, Re(\beta) > 0$and $H(z) \subset D$, we define

$$F(b,z,\beta) = \iint f(u,z,v)\, d\mu_b(u) d\mu_b(v) \quad (.2.5.2)$$

Modified Bessel function

Modified Bessel function of first kind of order p define by see (22-28)

$$I_p(z) \sum_{k=0}^{\infty} \frac{1}{k!\Gamma(p+1+k)} \frac{z}{2}^{2k+p} \quad z\in c$$

3. Main Result and proof :

Theorem1: Equivalence relation for double Dirichlet average of Modified Bessel function with the fractional derivative for $(k = x = 2)$ is

$$F(\mu, \mu'; z; \rho, \rho') = \frac{\Gamma(\rho+\rho')}{\Gamma\rho} (x-y)^{1-\rho-\rho'} D_{x-y}^{-\rho'} [I_p; (x)]\, (x-y)^{\rho-1} \quad (3.1)$$

Proof: Let us consider the double average for $(k = x = 2)$ of Modified Bessel function

$$F(I_p; \mu, \mu'; z; \rho, \rho') = \sum_{k=0}^{\infty} \frac{1}{k!\,\Gamma(p+1+k)} \int_0^1 \int_0^1 (u.z.v)^k\, dm_{\mu,\mu'}(u) dm_{\rho,\rho'}(v)$$
$$\sum_{k=0}^{\infty} \frac{1}{k!\,\Gamma(p+1+k)} = \int_0^1 \int_0^1 (u.z.v)^k dm_{\mu,\mu'}(u) dm_{\rho,\rho'}(v)$$

$Re(\mu) = 0, Re(\mu') = 0, Re(\rho) > 0, Re(\rho') > 0$ and

$$u.z.v = \sum_{i=1}^{2} \sum_{i=1}^{2} (u_i z_{ij} v_j) = \sum_{i=1}^{2} [u_i(z_{i1}v_1 + z_{i2}v_2)]$$
$$= [u_1 z_{11} v_1 + u_1 z_{12} v_2 + u_2 z_{21} v_1 + u_2 z_{22} v_2]$$

let $z_{11}=a, z_{12}=b, z_{21}=c, z_{22}=d$

and $\begin{cases} u_1=u, & u_2=1-u \\ v_1=v, & v_2=1-v \end{cases}$

Thus $z=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$$v.z.v=uva+ub(1-v)+(1-u)cv+(1-u)d(1-v)$$
$$=uv(a-b-c+d)+u(b-d)+v(c-d)+d$$
$$dm_{\mu,\mu'}(u)=\frac{\Gamma(\mu+\mu')}{\Gamma\mu\,\Gamma\mu'}u^{\mu-1}(1-u)^{\mu'-1}du$$
$$dm_{\rho,\rho'}(v)=\frac{\Gamma(\rho+\rho')}{\Gamma\rho\Gamma\rho'}v^{\rho-1}(1-v)^{\rho'-1}dv$$

Putting these value in eq (3.1)

$$F(I_p;\mu,\mu';z;\rho,\rho')=\frac{\Gamma(\mu+\mu')}{\Gamma\mu\,\Gamma\mu'}\frac{\Gamma(\rho+\rho')}{\Gamma\rho\Gamma\rho'}\sum_{k=0}^{\infty}\frac{1}{k!\,\Gamma(p+1+k)}\times\int_0^1\int_0^1 f[uv(a-b-c+d)+u(b-d)+v(c-d)+d]^k\, u^{\mu-1}(1-u)^{\mu'-1}v^{\rho-1}(1-v)^{\rho'-1}dudv$$

In order to obtained the fractional integral equivalent to the above integral,

Case –I: we assume $a=x, b=y$ and $c=d=0$ then

$$F(\mu,\mu';z;\rho,\rho')=\frac{\Gamma(\mu+\mu')}{\Gamma\mu\,\Gamma\mu'}\frac{\Gamma(\rho+\rho')}{\Gamma\rho\Gamma\rho'}\sum_{k=0}^{\infty}\frac{1}{k!\,\Gamma(p+1+k)}\times\int_0^1\int_0^1[uv(x-y)+uy]^k\,u^{\mu-1}(1-u)^{\mu'-1}v^{\rho-1}(1-v)^{\rho'-1}dudv$$

We use the pochammer symbols and consider the n be the power of the function in the above equations and using inspection method, we have

$$F(I_p;\mu,\mu';z;\rho,\rho')=\frac{\Gamma(\rho+\rho')}{\Gamma\rho\Gamma\rho'}\frac{(\mu)_n}{(\mu+\mu')_n}\sum_{k=0}^{\infty}\frac{1}{k!\,\Gamma(p+1+k)}\times\int_0^1[v(x-y)+y]^k\,v^{\rho-1}(1-v)^{\rho'-1}dv$$

Putting $v(x-y)=t$, we obtain

$$F(\mu,\mu';z;\rho,\rho')=\frac{\Gamma(\rho+\rho')}{\Gamma\rho\Gamma\rho'}\frac{(\mu)_n}{(\mu+\mu')_n}\sum_{k=0}^{\infty}\frac{1}{k!\,\Gamma(p+1+k)}\times\int_0^{x-y}[y+t]^n\left(\frac{t}{x-y}\right)^{\rho-1}\left(1-\frac{t}{x-y}\right)^{\rho'-1}\frac{dt}{(x-y)}$$

$$F(I_P;\mu,\mu';z;\rho,\rho')=\frac{\Gamma(\rho+\rho')}{\Gamma\rho\Gamma\rho'}\frac{(\mu)_n}{(\mu+\mu')_n}\sum_{k=0}^{\infty}\frac{1}{k!\,\Gamma(p+1+k)}(x-y)^{1-\rho-\rho'}\int_0^{x-y}[y+t]^k\,(t)^{\rho-1}(x-y-t)^{\rho'-1}dt$$

$$F(I_P;\mu,\mu';z;\rho,\rho')=\frac{\Gamma(\rho+\rho')}{\Gamma\rho\Gamma\rho'}\frac{(\mu)_n}{(\mu+\mu')_n}(x-y)^{1-\rho-\rho'}\int_0^{x-y}I_p;[y+t]\,(t)^{\rho-1}(x-y-t)^{\rho'-1}dt \tag{3.2}$$

Using definition of fractional integral,(2.4.1) we get

$$F(I_p;\mu,\mu';z;\rho,\rho')=\frac{\Gamma(\rho+\rho')}{\Gamma\rho}\frac{(\mu)_n}{(\mu+\mu')_n}(x-y)^{1-\rho-\rho'}D_{x-y}^{-\rho'}[I_p;(x)](x-y)^{\rho-1} \tag{3.3}$$

Case II: If we assume $a=c=x$; $b=d=y$ then the double Dirichlet average of any functions convert into single Dirichlet average of those functions. Thus we have

$$F(I_p;\mu,\mu';z;\rho,\rho')=\frac{\Gamma(\mu+\mu')}{\Gamma\mu\,\Gamma\mu'}\frac{\Gamma(\rho+\rho')}{\Gamma\rho\Gamma\rho'}\sum_{k=0}^{\infty}\frac{1}{k!\,\Gamma(p+1+k)}\times\int_0^1\int_0^1[v(x-y)+y]^k\,u^{\mu-1}(1-u)^{\mu'-1}v^{\rho-1}(1-v)^{\rho'-1}dudv$$

$$F(I_p;\mu,\mu';z;\rho,\rho')=\frac{\Gamma(\rho+\rho')}{\Gamma\rho\Gamma\rho'}\times\int_0^1 f[v(x-y)+y]^k\,v^{\rho-1}(1-v)^{\rho'-1}dv$$

Putting $v(x-y)=t$, we obtain

$$F(I_p;\mu,\mu';z;\rho,\rho')=\frac{\Gamma(\rho+\rho')}{\Gamma\rho\Gamma\rho'}\times\int_0^{x-y}\sum_{k=0}^{\infty}\frac{1}{k!\,\Gamma(p+1+k)}[y+t]^k\left(\frac{t}{x-y}\right)^{\rho-1}\left(1-\frac{t}{x-y}\right)^{\rho'-1}\frac{dt}{(x-y)}$$

$$F\left(I_p;\mu,\mu';z;\rho,\rho'\right)=\frac{\Gamma(\rho+\rho')}{\Gamma\rho\Gamma\rho'}(x-y)^{1-\rho-\rho'}\int_0^{x-y}\sum_{k=0}^{\infty}\frac{1}{k!\,\Gamma(p+1+k)}[y+t]^k\,(t)^{\rho-1}(x-y-t)^{\rho'-1}dt$$

$$F\left(I_p;\mu,\mu';z;\rho,\rho'\right)=\frac{\Gamma(\rho+\rho')}{\Gamma\rho\Gamma\rho'}(x-y)^{1-\rho-\rho'}\int_0^{x-y}[I_p;(x)]\,[y+t](t)^{\rho-1}(x-y-t)^{\rho'-1}dt \qquad (3.4)$$

Using definition of fractional derivative(2.4) we get

$$F\left(I_p;\mu,\mu';z;\rho,\rho'\right)=\frac{\Gamma(\rho+\rho')}{\Gamma\rho}(x-y)^{1-\rho-\rho'}D_{x-y}^{-\rho'}I_p;(x)\,(x-y)^{\rho-1} \qquad (3.5)$$

This is complete proof of (3.1)

4. Conclusion

In the present work It has been proved that any analytic function can be measured as Double Dirichlet average and connected with fractional derivative.

REFERENCES

[1] Al-Bassam, M.A.: Application of fractional calculus to differential equation of Hermite's type. Indian J. Pure Appl. Math. 16(9), (1985) 1009-1016

[2] Carlson, B.C., Special Function of Applied Mathematics, Academic Press, New York, 1977.

[3] Carlson, B.C., Appell's function F4 as a double average, SIAM J.Math. Anal.6 (1975), 960-965.

[4] Carlson, B.C., Hidden symmetries of special functions, SIAM Rev. 12 (1970), 332-345.

[5] Carlson, B.C., Dirichlet averages of x t log x, SIAM J.Math. Anal. 18(2) (1987), 550-565.

[6] Carlson, B.C., A connection between elementary functions and higher transcendental functions, SIAM J. Appl. Math. 17 (1969), 116-140.

[7] Deora, Y. and Banerji, P.K., Double Dirichlet average of ex using fractional derivatives, J. Fractional Calculus 3 (1993), 81-86.

[8] Deora, Y. and Banerji, P.K., Double Dirichlet average of xt and fractional derivatives, Rev.Tec.Ing.Univ. Zulia 16(2) (1993), 157-161.

[9] Deora, Y, and Banerji, P,K, An Application of Fractional Calculus to the solution of Euler-Darbox equation in terma of Dirichlet average J. of fractional Calculus vol.5, may (1994) 91-94.

[10]Erdelyi, A., Magnus, W., Oberhettinger, F. and Tricomi , F.G., Tables of Integral Transforms, Vol.2 McGraw-Hill, New York, 1954.

[11]Gupta,S.C. and Agrawal, B.M., Dirichlet average and fractional derivatives, J. Indian Acad.Math. 12(1) (1990), 103-115.

[12]Gupta,S.C. and Agrawal, Double Dirichlet average of ex using fractional derivatives, Ganita Sandesh 5 (1) (1991),47-52.

[13]Kilbas, A., A. and Kattuveetti, Anitha,: representations of dirichlet averages of generalized mittag-leffler function viafractional integrals and special functions, FCAA Vol.11(4) (2008) 471-492.

[14]Kiryalova, A.: Generalized Fractional Calculus and Applications, Pitman Research Notes in Mathematics Series No 301, Longman Scientific and Technical, Harlow, Essex, (1994), U.K.

[15]Kiryakova,V.: Multiindex Mittag-Leffler functions, related Gelfond- Leontiev operators and Laplace type integral transforms. Fract. Calc. Appl. Anal. 2, No 4 (1999), 445-462.

[16]Lorenzo,Carl. F., Hartley, Tom T.: Generalized Functions for the Functional Calculus, Nasa/tp-1999-209424/rev 1, p1-17.

[17]Mathai, A.M. and Saxena,R.K., The H-Function with Applications in Stastistics and other Disciplines, Wiley Halsted, New York, 1978.

[18]Miller, K. S., and Ross, B.: An Introduction to the Fractional Calculus and Fractional Differential Equations. John Wiley and Sons. Inc., (1993), New York.

[19]Prabhakar, T. R.: A singular integral equation with a generalized Mittag-Leffler function in the kernel, Yokohama Math. J., 19 (1971), 7-15

[20]Saxena,R.K., Mathai,A.M and Haubold, H.J., Unified fractional kinetic equation and a fractional diffusion equation, J. Astrophysics and Space Science 209 (2004) , 299-310.

[21]Saxena, R. K and Nishimoto, K.: N-fractional calculus of generalized Mittag-Leffler functions, J. Fract. Calc. 37(2010), pp. 43-52.

[22]Saxena, R. K and Nishimoto, K.: Further results on generalized Mittag-Leffler functions of fractional calculus, J. Fract. Calc. 39(2010), pp. 29-41.

[23] A. Baricz, ´ Generalized Bessel Functions of the First Kind, vol. 1994 of Lecture Notes in Mathematics, Springer, Berlin, Germany, 2010.

[24] A. Baricz, “Geometric properties of generalized Bessel func- ´ tions,” Publicationes Mathematicae Debrecen, vol. 73, no. 1-2, pp. 155–178, 2008.

[25] J. Choi, P. Agarwal, S. Mathur, and S. D. Purohit, “Certain new integral formulas involving the generalized Bessel functions,” Bulletin of the Korean Mathematical Society, vol. 51, no. 4, Article ID 9951003, 2014.

[26] P. Malik, S. R. Mondal, and A. Swaminathan, “Fractional integration of generalized bessel function of the first kind,” in Proceedings of the ASME International Design Engineering Technical Conferences and Computers and Information in Engineering Conference (IDETC/CIE ’11), pp. 409–418, Washington, DC, USA, August 2011.

[27] S. R. Mondal and K. S. Nisar, “Marichev-Saigo-Maeda fractional integration operators involving generalized Bessel functions,” Mathematical Problems in Engineering, vol. 2014, Article ID 274093, 11 pages, 2014.

[28] D. Baleanu, P. Agarwal, and S. D. Purohit, “Certain fractional integral formulas involving the product of generalized Bessel functions,” The Scientific World Journal, vol. 2013, Article ID 567132, 9 pages, 2013.

[29] G. N. Watson, A Treatise on the Theory of Bessel Functions, Cambridge University Press; The Macmillan, Cambridge, UK

[30]Saxena, R. K., Pogany, T. K., Ram, J. and Daiya, J.: Dirichlet Averages of Generalized Multi-index Mittag-Leffler Functions, AJM, Vol. 3, No. 4, (2010), 174-187.

[31]Sharma, Manoj and Jain, Renu,: Triple Dirichlet average of ex and fractional derivative, Applied Science Periodicals, (2008) 163-168

[31]Sharma, M., Jain, R., Sharma, K.: Dirichlet average of M-L Function and Fractional Derivative, Napier Indian Advanced Research Journal of Sciences, (2010) 05-06.

[32]Sharma, Manoj and Jain, Renu,: Double Dirichlet average of cos x and Fractional Derivative, Ganita Sandesh, India, (2007) 107-110.

[33]Sharma, Manoj and Jain, Renu,: Double Dirichlet average of and Fractional Derivative, Journal of Indian Acad. Math, India, (2006) 337-342.

[34]Sharma, Manoj and Jain, Renu, Dirichlet Average of ﻁ◌□ℎﭗ and Fractional Derivatie, J Indian Acad. Math.Vol.2, No. 1(2007). P17-22.

[35] Srivastava, H. M., Goyal, S. P.and Jain, R. M.: Fractional integral operators involving a general class of polynomials, J. Math. Anal. Appl. 148 (1990), 87-100