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An Application of Fractional Calculus in RLC Circuit By Using Laplace Transform

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Abstract: An Italian scientist Caputo introduce a Caputo fractional derivative in 1967, a lots of work has been done by using this Caputo fractional differential equation in RLC. In this paper we are also using a Caputo fractional differential equation operator .We have define a LCR circuit viz. inductance *L*, capacitance *C*, and resistance *R* , in a closed form in term of the three parameters Mittag –laffler functions . Three Parameters Mittag –Leffler was introduced by Prabhakar in 1971.

Keywords: RLC circuit, Mittag-Laffler function and its extension, Mathieu series, fractional derivative.

1. Introduction: First we will introduce the calculus, which was discovered by Isaac Newton and Gottfried Wilhelm in seventeenth century. Calculus is a type of Mathematics which is deal with rate of change, for example the velocity of object and the acceleration of object. When Leibnitz introduced the calculus in 1695 at that time L' hospital asked query to him what will happened if n will take $1/2$ value in $\frac{d^n y}{dx^n}$ .After two year in 1697 Leibnitz replied him by wrote a letter "One day useful consequence will be drawn from this fact" The credit of organizing first international conference on fractional calculus goes to Bertram Ross was held in New Heaven (U.S.A) in 1974. After that more than three hundred years have been passed with these passing years, several aspects of fractional calculus have been developed and studied. In other words, the fractional calculus operator deal with integrals and derivatives of arbitrary order.

Mathematicians and Physicists found that the fractional calculus operators is very useful in a various fields such as quantitative biology, electro chemistry, scattering theory, transport theory, probability, elasticity, control theory, potential theory, signal processing, image processing, diffusion theory, kinetic theory, heat transfer theory and circuit theory etc.

The accurate use of a derivative of non-integer order is due to the French mathematician S. F. Lacroix [14] in 1819.he expressed the derivative of non-integer order ½ in terms of Legendre's factorial symbol Γ.

$$\Gamma(a) = \int_0^\infty t^{\alpha-1} e^{-t} dt$$

Starting, with a function $y = x^m$, Lacroix expressed it as follows

$$\frac{d^n y}{dx^n} = \frac{m!}{(m-n)!} x^{m-n} = \frac{\Gamma(m+1)}{\Gamma(m-n+1)} x^{m-n}$$

Replacing n with $\frac{1}{2}$ and putting $m = 1$, he obtained the derivative of order $\frac{1}{2}$ of the function x.

$$\frac{d^{1/2} y}{dx^{1/2}} = \frac{\Gamma(2)}{\Gamma(3/2)} x^{1/2} = \frac{2}{\sqrt{\pi}} \sqrt{x}$$

In 1822, J. B. J. Fourier made the following integral representation

$$\frac{d^u f(x)}{dx^u} = \frac{1}{2\pi} \int_{-\infty}^{+\infty} f(\alpha) d\alpha \int_{-\infty}^{+\infty} p^u \cos\left(px - p\alpha + \frac{u\pi}{2}\right) dp$$

where the number u was regarded as any quantity whatever, positive or negative.

The credit of first application of fractional calculus goes to Abel's [12], he employed it in the solution of an integral equation, which emerged in the formulation of the tautochrone problem of finding the shape of a frictionless wire lying in a vertical plane such that the time of slide of a bead placed on the wire to the lowest point of the wire is the same regardless of position of the bead on the wire.

2. The Mittag-Leffler Function:

The Mittag-Leffler function introduced by Mittag-Leffler [18] in 1903, is defined as

$$E_\alpha(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(\alpha k+1)}, (\alpha \in C, Re(\alpha) > 0) \quad (1)$$

Generalization of the Mittag-Leffler function is given by Wiman [20] in 1905, defined as

$$E_{\alpha,\beta}(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(\alpha k+\beta)}, (\alpha, \beta \in C, Re(\alpha) > 0, Re(\beta) > 0) \quad (2)$$

Prabhakar [19] introduced generalization of (2) in 1971 in the form

$$E_{\alpha,\beta}^{\gamma}(x) = \sum_{k=0}^{\infty} \frac{(\gamma)_k x^k}{\Gamma(\alpha k+\beta)\, k!}, (\alpha, \beta, \gamma \in C, Re(\alpha) > 0), Re(\beta) > 0 \quad (3)$$

Where $(\gamma)_k$ is the Pochammer symbol

It is an entire function with $\rho = [Re(v)]^{-1}$.

For $\gamma = 1$,this function coincides with (2), while for $\gamma = \beta = 1$with;

$$E_{\alpha,\beta}^{1}(x) = E_{\alpha,\beta}(x), E_{\alpha,1}^{1}(x) = E_\alpha(x) \quad (4)$$

We also have

$$\phi(\beta, \gamma; x) = {}_1F_1(\beta, \gamma; x) = \Gamma\gamma\, E_{1,\gamma}^{\beta}(x) \quad (5)$$

$$(x) = \frac{1}{\Gamma\gamma} H_{1,2}^{1,1}\left[-x \middle| \begin{matrix} (1-\gamma, 1) \\ (0,1), (1-\beta, \alpha) \end{matrix}\right], Re(\alpha) > 0; \alpha, \beta, \gamma \in C \quad (6)$$

For $\gamma = 1$(6) gives rise to the following result for the generalized Mittag-Leffler function.

$$E_{\alpha,\beta}(x) = H_{1,2}^{1,1}\left[-x \middle| \begin{matrix} (0,1) \\ (0,1), (1-\beta, \alpha) \end{matrix}\right], Re(\alpha) > 0; \alpha, \beta \in C \quad (7)$$

If we further take $\beta = 1$in (7)we find that

$$E_{\alpha}(x) = H_{1,2}^{1,1}\left[-x \middle| \begin{matrix} (0,1) \\ (0,1), (0, \alpha) \end{matrix}\right], Re(\alpha) > 0; \alpha \in C \quad (8)$$

Reimann-Liouville operator, Modified Reimann- Liouville fractional derivative, Caputo fractional derivative, Wely fractional operator, Tuan and Saigo Fractional Operators they all mathematician given the definition of fractional derivative The Reimann –Liouville fractional derivative of constant and Caputo fractional derivative of constant are not equal to each other .The Caputo fractional derivative of constant is Zero.

3. Caputo Fractional Derivative:

The Caputo fractional derivative of order $\alpha > 0$ is introduced by Caputo [17] in the form

(if $m-1 < \alpha \le m, Re(\alpha) > 0, m \in N$)

$${}_a^cD_t^\alpha f(t) = I^{m-\alpha} D^m f(t)$$

or

$${}_a^cD_t^\alpha f(t) = \frac{1}{\Gamma(m-\alpha)} \int_0^t \frac{f^m(\tau)d\tau}{(t-\tau)^{\alpha+1-m}}, t>0$$

$$= \frac{d^m f(t)}{dt^m}, \text{if}\alpha = m \quad (9)$$

Where $\frac{d^m f(t)}{dt^m}$ is the m-th derivative of order m of the function $f(t)$ with respect to*t*.

Or

$${}_0^cD_x^\alpha f(x) = \frac{1}{\Gamma(1-\alpha)} \int_0^x \frac{f'^{(t)}}{(x-t)^\alpha} dt,$$

where $0 < \alpha < 1$) (10)

According to this definition

${}_a^cD_t^\alpha A = 0$ *f*(t) = A = Constant

This is Caputo's fractional derivative of a constant is zero. The Laplace transform of Caputo derivative is representation of

$$L[{}_a^cD_t^\alpha f(t)] = s^\alpha F(s) - \sum_{k=1}^{n-1} f^k(0) s^{\alpha-k-1} \quad (11)$$

We see that from the equation (11) the representation of the Caputo derivative in Laplace domain using the initial condition $f^k(0)$ where k is integer. When the initial condition is zero then the equation (11)converted into

$$L[{}_a^cD_t^\alpha f(t)] = s^\alpha F(s)$$

4. Mathieu Type series

In this section establish the relation in Advanced Mathieu type series and Generalized Mathieu type series

For equation (7) $d_{(k+1)} = d_k$ where $d = \{d_{(k+1)}\}_{k=0}^{\infty}$ $= \{d_1 d_{2...}\}(\log_{k\to\infty} d_k = \infty)$ is such that the infinite series $\sum_{k=0}^{\infty} \frac{1}{d_k{}^{\mu\rho-\sigma}}$ is convergent. $(k+1)! = k!$ $(l+1) = 1$, $\frac{(a_1)_k \cdots (a_p)_k}{(b_1)_k \cdots (b_q)_k} = (\tau)_k$

Generalized Mathieu type series defined by Tomovski and Mehrez [22]in 2017 in the form

$$s_{\mu,\tau}^{\rho,\sigma}(l,d,u) = s_{\mu,\tau}^{\rho,\sigma}(l,\{d_k\}_{k=1}^{\infty};u) = \sum_{k=1}^{\infty} \frac{2d_k^{\sigma}(\tau)_k}{(d_k^{\rho}+l^2)} \frac{u^k}{k!}$$

$(l, d, \rho, \mu \epsilon R^+;$ lul ≤ 1 (12)

Mathieu series introduced by Tomovski and pogany [23] in 2011,define as

For equation (12) $d_k = k, \rho = 2, \sigma = 1 = \tau$ *and* μ *replace for* $\mu + 1$ then we obtained the Mathieu series defined by Tomovski and pogany

$$S_{\mu+1,1}^{2,1}(l,k,u = S_\mu(l,u) = \sum_{k=1}^{\infty} \frac{2k\, u^k}{(k^2+l^2)^{\mu+1}} \quad (l,\mu \epsilon R^+; \text{lul} \leq 1) \tag{13}$$

5. Definition

Polynomial Function:

Let $f(t)$ be the polynomial function which can be mathematically expressed as follows:

$$f(t) = (at^k + bt^{k-1} + ct^{k-2} + \cdots d) \tag{14}$$

Where k > 0

Unit Parabolic Function:

The unit parabolic function defined as follows:

$$f(t) = \begin{cases} 0 & for\ t < 0 \\ \frac{t^2}{2} & for\ t \geq 0 \end{cases} \tag{15}$$

And its Laplace transform is $\frac{1}{s^3}$.

6. *RLC* Electrical Circuit

We have considered a RLC electrical circuit with a capacitor and an inductor, connected in parallel and this set is connected in series with a resistor and a voltage. The capacitance *C*, the inductance *L*, and the resistor *R*, are consider positive constants and $f(t)$ is the polynomial function. Earlier, Soubhia, Camargo and Rubens [15], consider the $\psi(t)$ is Heaviside function in their paper and Recently Farman Ali , Manoj Sharma, and Renu jain [24] consider, $\theta(t)$ is the unit ramp function in their paper.

The following equations associated with a three elements of *RLC* electrical circuit as under:

The voltage drop

$$U_L(t) = L\frac{d}{dx}I(t), \quad \text{across an inducter;}$$

The voltage drop

$$U_c(t) = \frac{1}{C}\int_0^t I(\xi)d\xi, \text{across a capacitor}$$

The voltage drop

$$U_c(t) = \frac{1}{C}\int_0^t I(\xi)d\xi, \text{across a capacitor}$$

where $I(t)$ is the current.

Applying the Kirchhoff's voltage law and above equations associated with the three elements. The non-homogeneous second order ordinary differential equation can be written as

$$RC\frac{d^2}{dt^2}U_C(t) + \frac{d}{dt}U_C(t) + \frac{R}{L}U_C(t) = \frac{d}{dt}\sum_{k=1}^{\infty}\frac{2k\, t^k}{(k^2+l^2)^{\mu+1}} \tag{16}$$

Here, $U_c(t)$ is the voltage on in the capacitor, this is the same on the inductor as we can see in following figure (1), because they are connected in parallel.

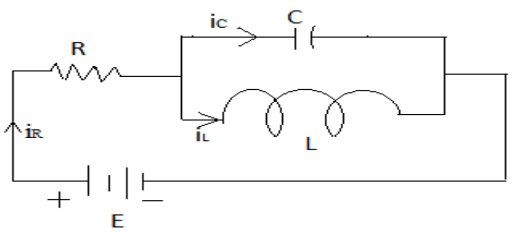

Fig. 1: Three elements LCR electrical Circuit

Again, for current on the inductor, we get other non-homogeneous second order ordinary differential equations,

$$RLC\frac{d^2}{dt^2}i_L(t)+L\frac{d}{dt}i_L(t)+Ri_L(t)$$
$$=\sum_{k=1}^{\infty}\frac{2\mathrm{k}\,\mathrm{t}^{\mathrm{k}}}{(\mathrm{k}^2+\mathrm{l}^2)^{\mu+1}} \qquad (17)$$

Now, using the above voltage drop equation across a capacitor for the inductor, therefore these two non-homogeneous second order ordinary differential equations correspond to integro-differential equations as follows.

$$R\frac{d}{dt}i_C(t)+\frac{1}{C}i_C(t)+\frac{R}{LC}\int_0^t i_C(\xi)d\xi,$$
$$=\frac{d}{dt}\sum_{k=1}^{\infty}\frac{2\mathrm{k}\,\mathrm{t}^{\mathrm{k}}}{(\mathrm{k}^2+\mathrm{l}^2)^{\mu+1}} \qquad (18)$$

And

$$RC\frac{d}{dt}U_L(t)+U_L(t)+\frac{R}{L}\int_0^t U_L(\xi)d\xi,$$
$$=kt^{k-1}\sum_{k=1}^{\infty}\frac{2\mathrm{k}}{(\mathrm{k}^2+\mathrm{l}^2)^{\mu+1}} \qquad (19)$$

We observe that, both integro-differential equations are in the similar form. Here we consider only the first one i.e. (18). The above integro-differential equation (18) solved by technique of Laplace transform with initial condition $i_C(0)=0$ in classical methodology and the solution found in terms of an exponential function [21].

7. Fractional Integro-Differential Equation

In this, we generalize ordinary differential equations (18) into the fractional form associated with a current on the capacitor, which is known as fractional integro-differential equation:

$$R\frac{d^\alpha}{dt^\alpha}i_C(t)+\frac{1}{C}i_C(t)+\frac{R}{LC}\frac{1}{\Gamma\alpha}\int_0^t(t-\xi)^{\alpha-1}i_C(\xi)d\xi=\frac{d}{dt}\sum_{k=1}^{\infty}\frac{2\mathrm{k}\,\mathrm{t}^{\mathrm{k}}}{(\mathrm{k}^2+\mathrm{l}^2)^{\mu+1}} \qquad (20)$$

With $0<\alpha\le 1$, and the fractional derivative is taken in the Caputo sense, where (t^k) is the polynomial function. In this case, $i_C(t)$can be interpreted as a Green's function because the second member is polynomial function. Let us consider $i_C(0)=0$, i.e., the initial current on the capacitor is zero. We obtained the result of equation (18) is same as fractional integro-differential equation (20) associated with the RLC electrical circuit, if $\alpha = 1$. This replacement can be useful in discussing the corresponding numerical problem, for a particular value of the parameter, because the solution is shown in terms of a closed expression.

Use the Laplace integral transform, to solve this fractional integro-differential equation:

$$L[i_C(t)]=F(s)=\int_0^\infty e^{-st}i_C(t)\,dt$$

with $Re(s)>0$, and the following algebraic equation, $\sum_{k=1}^{\infty}\frac{2\mathrm{k}}{(\mathrm{k}^2+\mathrm{l}^2)^{\mu+1}}=R'$

$$Rs^\alpha F(s)+\frac{F(s)}{C}+\frac{R}{LC}\frac{F(s)}{s^\alpha}=\frac{\Gamma k}{s^k}\;R'$$

The solution is given by

$$F(s)\left(Rs^\alpha+\frac{1}{C}+\frac{R}{Lc\,s^\alpha}\right)=\frac{\Gamma k}{s^k}\;R'$$

$$F(s)=\frac{\Gamma kR'}{R}\frac{s^{\alpha-k}}{(s^{2\alpha}+b+as^\alpha)}$$

Where, $a=1/RC$ and $b=1/LC$ are the positive parameters.$\frac{1}{R''}=\frac{\Gamma k}{R}R'$

Taking inverse Laplace transform of both sides, we get

$$i_C(t)=\frac{1}{R''}L^{-1}\left[\frac{s^{\alpha-k+1-1}}{s^{2\alpha}+as^\alpha+b}\right]$$

Applying the formula [30]

$$L^{-1}\left[\frac{s^{\rho-1}}{s^\alpha+As^\beta+B}\right]$$
$$=t^{\alpha-\rho}\sum_{r=0}^{\infty}(-A)^r t^{(\alpha-\beta)r}E_{\alpha,\alpha+1-\rho+(\alpha-\beta)r}^{r+1}(-Bt^\alpha)$$
$$\text{Valid for }\left|\frac{As^\beta}{s^\alpha+B}\right|<1\text{ and }\alpha\ge\beta,$$

We can write

$$i_C(t) = \frac{t^{\alpha+k-1}}{R''}\sum_{r=0}^{\infty}(-a)^r t^{\alpha r} E_{2\alpha\alpha+k+\alpha r}^{r+1}(-bt^{2\alpha}) \sum_{k=1}^{\infty}\frac{2\mathrm{k}\,\mathrm{t}^{\mathrm{k}}}{(\mathrm{k}^2+\mathrm{l}^2)^{\mu+1}} \quad (21)$$

Where $E_{\mu,v}^{\rho}(t)$ is known as the three parameters Mittag-Leffler functions and t^k is the polynomial function.

This completes the analysis.

Corollary :1

If we take k =1 and $\sum_{k=1}^{\infty}\frac{2\mathrm{k}}{(\mathrm{k}^2+\mathrm{l}^2)^{\mu+1}}=1$ in equation (19) then its convert into Ali's Paper result "An Application of Fractional Calculus in Electrical Engineering"

$$i_C(t) = \frac{t^{\alpha}}{R'}\sum_{r=0}^{\infty}(-a)^r t^{\alpha r} E_{2\alpha,\alpha+1+\alpha r}^{r+1}(-bt^{2\alpha})$$

8. Conclusion

In this paper we propose new results for series in three – Parameter Mittag-Laffler functions, Presenting, them explicitly in term of the two-parameters Mittag-Laffler Function. A natural continuation of this paper is to discuss the gamma distribution whose density function is known. The possible results, we obtain a closed form to the solution of the fractional integro- differential equation associated with a particular RLC electrical circuit , in terms of the three-parameter Mittag-Leffer function. Our main result is interesting with respect to simplifying several other results, for i.e. as one can see in [4] where we discussed the fractional telegraph equation, and in [3], where the anomalous diffusion was presented. The results in both papers are given in terms of the three- parameter Mittag-Leffer function

REFERENCES

[1] R.P. Agarwal, A propos d'une note de M. Pierre Humbert. *C.R. Acad. Sci. Paris*, 236 (1953), 2031-2032.

[2] R. Figueiredo Camargo, *Fractional Calculus and Applications*(in Portuguese). Doctoral Thesis, Unicamp, Campinas, 2009.

[3] R. Figueiredo Camargo, E. Capelas de Oliveira and J. Vaz Jr., On the anomalous diffusion and the fractional generalized Langevin equation. *J. Math. Physics* 50, (2009) 123518 (publ.online Dec. 2009).

[4]R. Figueiredo Camargo, E. Capelas de Oliveira and J. Vaz Jr., On the generalized Mittag-Leffer function and its application in a fractional telegraph equation. Submitted for publication, 2009.

[5]R. Figueiredo Camargo, A.O. Chiacchio and E. Capelas de Oliveira, Di®erentiation to fractional orders and the telegraph equation. *J. Math. Phys.* 49 (2008), 033505.

[6] R. Figueiredo Camargo, A.O. Chiacchio, R. Charnet and E. Capelas de Oliveira, Solution of the fractional Langevin equation and the Mittag-Leff er functions. *J. Math. Phys.* 50 (2009) 063507.

[7] Ch. Fox, The *G*- and *H*- functions as symmetrical Fourier kernels.*Trans. Amer. Math. Soc.*, 98 (1961), 395-429.

[8] R.Hilfer (Ed.), *Applications of Fractional Calculus in Physics*. World Scienti¯c, Singapore (2000).

[9] R. Charnet and E. Capelas de Oliveira, On the product of two gamma distribution and the Mittag-Laffler function .Submitted.

[10] R.Figueiredo Camargo, RCharnet andE.Capelas de oliveira ,on some fractional Green's function j.math.phys. 50(2009) 043514.

[11]V. Kiryakova, The multi-index Mittag-Le²er functions as important class of special functions of fractional calculus. *Com- puters and Math. with Appl.* 59, No5 (2010), 1885-1895

[12] N. H. Abel: Solution De Quelques Problems a L'aide D'integrals Definites. Oeuvres Completes, Vol. 1 (1881), Grondahl Christiania, Norway, 16-18.

[13] F. Mainardi, On the initial value problem for the fractional diffusion- wave equation. In: S. Rionero and T. Ruggeri (Eds.), *Waves and Stability in Continuous Media*, World Scienti¯c, Singapore (1994), 246-251.

[14] S. F.,Lacroix: Traite du Calcul Differentiel et du Calcul integral, 2nd ed. Vol. 3, Courcier, Paris. 409-410.

[15]Ana L. Soubhia, Rubens F. Camargo, Edmundo C. de Oliveira and Jayme Vaz Jr.,Theorem for series in three-parameter Mittag- leffler function, *Fract. Calc. Appl. Anal.* 13, No 1 (2010), 8-20.

[16] R.L. Magin and M. Ovadia, Modeling the cardiac tissue electrode interface using fractional calculus. *J. Vibration and Control* 14 (2008), 1431-1442.

[17] M. Caputo: Linear Models of Dissipation Whose Q is Almost Frequency Independent, II. Geophysical J. of the Royal Astronomical Soc., 13 (1967), 529-539 (Reprinted in: Fract. Calc. Appl. Anal., Vol. 11 No. 1 (2008) 3-14.

[18] G.M. Mittag- Leffler: Sur la nouvelle function (x) C. R. Acad. Sci., Paris (Ser. II), 137 (1903), 554-558.

[19] T.R. Prabhakar: A Singular Integral Equation with a Generalized Mittag-Leffler Function in the Kernel, Yokohama Math. J., Vol. 19 (1971) 7-15.

[20] A.Wiman: Ueber den Fundamentalsatz in der Theorie der Funkilionen , Acta. Math., Vol. 29 (1905), 191-201.

[21]L. Boyadjiev, S. L. Kalla, and H. G. Khajah: Analytical and Numerical Treatment of a Fractional Integro-Differential Equation of Volterra-Type, Math. comput. Modelling, Vol. 25, No.12 (1997) 1-9.

[22]Tomovski ,z, Mehrez,M some families of generalized Mathieu –type power series .Appl. Anal discrete math. 20(4),973-986(2017)

[23] Tomovski, and pogany , TK integral expressions for Mathieu-type power series and the Butzer-Flocke-Hauss function. Fract Calc.Appl 14(4), 623-634(2011)

[24] Farman ali , Sharma, and Jain ,An Application of Fractional Calculus Electrical Engineering ,Adv .Eng. Tec .Appl. 5, No.2, 41-45 (2016)