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Solution of Cauchy-Type Problem for the Fractional Diffusion Equation in Terms of M-series

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Abstract: We deal with the new Unified fractional diffusion model with non-linear function and obtained the solution of fractional diffusion equation by using a Fractional Fourier transform, Classical Laplace transform. This problem has derived the result in the form of theorem in terms of M-series.

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1. Introduction

Fractional calculus is an ordinary differentiation of non-integer case as natural. Fractional calculus is defined by many mathematician- Riemann-Liouville, Caputo etc. Fractional calculus contains the different types of models as fractional diffusion model, wave model etc. This model is very useful to solving the problem of Astrophysical Science like- the formation of star, Galaxies etc. We have studied the fractional diffusion equation related papers is given by Haubold, Mathai and Saxena [4] and obtained the solution in terms of H-function, next Chaurasia and Singh [3] obtained the solution of diffusion equation by using the Integral transform.
In the same way, we have studied the fractional diffusion model with non-linear function means product function and derived the solution in terms of M-series.

2. Mathematical definition

2.1 Riemann-Liouville fractional integral [8,10,11]

The right-sided Riemann-Liouville fractional integral of order α is defined as follows

$$ {}_aD_x^{-\alpha} f(x) = \frac{1}{\Gamma\alpha}\int_a^x (x-\tau)^{\alpha-1} f(\tau)d\tau \quad ,x>a, Re(\alpha)>0 \tag{1}$$

2.2 Riemann-Liouville fractional derivative [8,10,11]:-

The right-sided Riemann-Liouville fractional derivative of order α is defined as follows

$$ {}_aD_x^{\alpha} f(x) = [\left(\frac{d}{dx}\right)^n \{\frac{1}{\Gamma(n-\alpha)}\int_a^x (x-\tau)^{n-\alpha-1} f(\tau)d\tau\}] \tag{2}$$

Where $Re(\alpha)>0\,, n=[Re(\alpha)+1], \alpha =$Integral part of number α

2.3 Caputo derivative:

The Caputo fractional derivative is given by Caputo [2] of order $\alpha>0$ is defined as

$$ {}_a^cD_x^{\alpha} f(t) = \begin{bmatrix} \frac{1}{\Gamma(n-\alpha)}\int_a^x \frac{f^n(\tau)d\tau}{(x-\tau)^{\alpha+1-n}} & if\ n-1<\alpha\le n \\ \frac{d^n f(t)}{dt^n} & if\ \alpha=n \end{bmatrix} \tag{3}$$

Where $Re(\alpha)>0, n\in N$

2.4 Laplace transform:

This Transform is mathematically expressed as follow

$L\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t)dt \quad , \quad t > 0$ (4)

2.5 Inverse Laplace transform:-

Let given a function$F(s)$, the inverse Laplace transform of F denoted by $L^{-1}[F]$ is that function f whose Laplace transform is F

More succinctly $f(t) = L^{-1}[F(s)]$ (5)

2.6 Fourier transform:

This transform is mathematically expressed for real axis R as follow

$F\{u(x)\} = u^*(k) = \int_{-\infty}^{\infty} e^{ikx} u(x)dx \quad , k \in R$ (6)

2.7 Inverse Fourier transform:-

This transform is mathematically expressed as follows

$F^{-1}\{u^*(k)\} = u(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ikx} u^*(k)dk \quad , x \in R$
(7)

2.8 Fractional Fourier transform:

The Fractional Fourier transform u_α^* of the order α, ($0<\alpha \leq 1$) is defined as

$F_\alpha[u(x)] = u_\alpha^*(k) = \int_{-\infty}^{\infty} e_\alpha(k,x)u(x)dx$ (8)

Where $k \in R,\ u \in \emptyset(R),\ e_\alpha(k,x) = \begin{Bmatrix} e^{-i|k|\frac{x}{\alpha}} & k \leq 0 \\ e^{i|k|\frac{x}{\alpha}} & k > 0 \end{Bmatrix}$

If $\alpha = 1$ this equation (8) is reduce to the Kernel, relation between fractional Fourier transform and the Classical transform is given by

$u_\alpha^*(k) = F_\alpha[u(x)] = F_1[u(x)] = u^*(w)$ (9)

If $F_\alpha[u(x)] = F_1[u(x)] = \emptyset(w)$ … (10)

Then we get $u(x) = F_\alpha^{-1}[u_\alpha^*(k)] = F_1^{-1}[\emptyset(w)]$ (11)

Where $W = \begin{Bmatrix} -|k|^{\frac{1}{\alpha}} & k \leq 0 \\ |k|^{\frac{1}{\alpha}} & k > 0 \end{Bmatrix}$

2.9 Fractional Fourier transform related theorem:

Theorem 2.9.1: If $0 \leq \alpha < 1$ and $u^n(x) \in \emptyset(R)$ then already existing the result [1] has been given as

$F_\alpha[u^n(x)] = (-isignk|k|^{\frac{1}{\alpha}})^n\, u_\alpha^*(k) \quad ,k \in R$ (12)

Theorem 2.9.2: If $0 \leq \alpha < 1$ and $u(x), v(x) \in \emptyset(R)$ then already existing the result [9] has been given as

$F_\alpha[(u * v)x] = u_\alpha^*(k)v_\alpha^*(k)$ (13)

Here

$F_\alpha[u(x)] = u_\alpha^*(k),\ F_\alpha[v(x)] = v_\alpha^*(k)$ and $(u * v)x = \int_{-\infty}^{\infty} u(x-\rho)v(\rho)d\rho$

2.10 M-series: The M-series was introduced by Sharma [12]

${}_pM_q^\alpha(a_1 \ldots a_p; b_1 \ldots b_q; t) = \sum_{k=0}^{\infty} \frac{(a_1)_k \cdots (a_p)_k}{(b_1)_k \cdots (b_q)_k} \frac{t^k}{\Gamma(\alpha k+1)}$ … (14)

Here $\alpha, \beta \in C, Re(\alpha) > 0, (a_j)_k\ (b_j)_k$ are pochammer symbols.

2.11 Generalized M-series:

The Generalized M-series were introduced by Sharma and Jain [13]

${}_pM_q^{\alpha,\beta}(a_1 \ldots a_p; b_1 \ldots b_q; t) = \sum_{k=0}^{\infty} \frac{(a_1)_k \cdots (a_p)_k}{(b_1)_k \cdots (b_q)_k} \frac{t^k}{\Gamma(\alpha k+\beta)}$ …
(15)

Here $\alpha, \beta \in C, Re(\alpha) > 0, Re(\beta) > 0;\ (a_j)_k\ (b_j)_k$ are pochammer symbols.

2.12 Product Function:

This product function is multiplied by t and double sided exponential function $e^{-a|x|}$. It is defined by

$\Phi(x,t) = te^{-a|x|}$ With $a > 0$ (16)

3. Laplace transform of Caputo fractional derivative:

The Laplace transform of Caputo fractional derivative is given as

L $\{ {}_{0}^{c}D_t^{\alpha} f(t)\} = S^{\alpha} f(s) - \sum_{r=0}^{m-1} S^{\alpha-r-1} f^{r}(x, 0+)$, $(m-1 < \alpha \leq m)$ … (17)

Hilfer [5] extended the Riemann-Liouville fractional order derivative and Caputo fractional order derivative of two parameter of order $0 < \alpha < 1$ and $0 < \beta \leq 1$ in the form

$${}_{0}D_{a+}^{\alpha,\beta} f(t) = (I_{a+}^{\beta(1-\alpha)} \frac{d}{dt}\left(I_{a+}^{(1-\beta)(1-\alpha)} f(t)\right) \qquad (18)$$

More information of this operator are found in Tomovski et al [15]

The Laplace transform of this equation is

$$L\left\{{}_{0}D_{a+}^{\alpha,\beta} f(t)\right\} = S^{\alpha} f(s) - S^{\beta(\alpha-1)} I_{0+}^{(1-\beta)(1-\alpha)} f(x, 0+) \ , (0 < \alpha < 1) \qquad (19)$$

Where $I_{0+}^{(1-\beta)(1-\alpha)}$ is the initial value term. It is a Riemann-liouville fractional order integral operator of order $(1-\beta)(1-\alpha)$ and limit as $t \to 0+$

The fractional order derivative operator for extending the time-space diffusion equation

$$D_{\beta}^{\alpha} u(x) = (1-\beta) D_{+}^{\alpha} u(x) - \beta D_{-}^{\alpha} u(x) \qquad (20)$$

Here $D_{+}^{\alpha} u(x) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_{-\infty}^{x} (x-\tau)^{\alpha-1} u(\tau) d\tau$

$$D_{-}^{\alpha} u(x) = -\frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_{x}^{\infty} (x-\tau)^{\alpha-1} u(\tau) d\tau$$

This equation form [6] is very useful for solving the fractional diffusion equation by using the fractional Fourier transform

$$F_{\alpha}\left[D_{\beta}^{\alpha} u(x)\right] = (-iC_{\alpha} k) F_{\alpha}[u(x)] \quad k \in R \qquad (21)$$

Where $0 < \alpha \leq 1$, $\beta =$ any value, $u(x) \in \emptyset(R)$,

$$C_{\alpha} = \sin\left(\frac{\alpha\pi}{2}\right) + i sign k \ (1-2\beta) \cos\left(\frac{\alpha\pi}{2}\right)$$

4. New Unified fractional diffusion equations with non-linear function [14]:-

In this section, we derived the solution of following Cauchy-type problem for the fractional diffusion equation

$${}_{0}D_{t}^{\alpha,\beta} N(x,t) = \delta^{1/k} D_{\eta}^{\gamma+1} N(x,t) + \Phi(x,t) \ , x \in R, t > 0, \delta > 0, k > 0 \qquad (22)$$

Under the initial condition

$$I_{0+}^{(1-\beta)(1-\alpha)} N(x, 0+) = f(x) \qquad (23)$$

Here ${}_{0}D_{t}^{\alpha,\beta}$ is the extended Riemann-liouville fractional derivative operator , $\delta^{1/k}$ is the diffusion constant and its value is $\delta = \sum_{k=0}^{\infty} \frac{(a_1)_k \cdots (a_p)_k}{(b_1)_k \cdots (b_q)_k}$, $D_{\eta}^{\gamma+1}$ is the extended Riemann-Liouville space fractional derivative of order $\gamma + 1$, $I_{0+}^{(1-\beta)(1-\alpha)}$ is the Riemann-Liouville fractional integral operator of order $(1-\beta)(1-\alpha)$

Theorem 4.1: If $f(x) \in \emptyset(R), 0 < \alpha < 1, 0 < \beta \leq 1, 0 < \gamma \leq 1$ and for every value of $\eta \in R$

Consider the Cauchy –type problem for the fractional diffusion equation

$${}_{0}D_{t}^{\alpha,\beta} N(x,t) = \delta^{1/k} D_{\eta}^{\gamma+1} N(x,t) + \Phi(x,t) \ , x \in R, t > 0, \delta > 0, k > 0$$

Under the initial condition

$$I_{0+}^{(1-\beta)(1-\alpha)} N(x, 0+) = f(x)$$

Then we get

$$N(x,t) = \int_{-\infty}^{\infty} G_1(x-\tau, t) f(\tau) d\tau + \int_{0}^{t} (t-\xi)^{\alpha-1} d\xi \int_{-\infty}^{\infty} G_2(x-\tau, t-\xi) \Phi(\tau, \xi) d\tau \qquad (24)$$

Where

$$G_1(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ikx} t^{\alpha-\beta(\alpha-1)-1} {}_{p}M_{q}^{\alpha,\alpha-\beta(\alpha-1)}\left(-iC_{\gamma+1} k t^{\alpha}\right) dk$$

$$G_2(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ikx} {}_{p}M_{q}^{\alpha,\alpha}\left(-iC_{\gamma+1} k t^{\alpha}\right) dk$$

Proof:-Cauchy type problem for the fractional diffusion equation

$${}_{0}D_{t}^{\alpha,\beta} N(x,t) = \delta^{1/k} D_{\eta}^{\gamma+1} N(x,t) + \Phi(x,t)$$

Taking the Laplace transform of the both side and using initial condition

$$S^{\alpha} N(x,s) - S^{\beta(\alpha-1)} f(x) = \delta^{1/k} D_{\eta}^{\gamma+1} N(x,s) + \Phi(x,s) \ (25)$$

Taking the Fourier transform of the both side

$$S^{\alpha} N^*_{\gamma+1}(k,s) - S^{\beta(\alpha-1)} f^*_{\gamma+1}(k) = \left(-i\delta^{1/k} C_{\gamma+1} k\right) N^*_{\gamma+1}(k,s) + \Phi^*_{\gamma+1}(k,s)$$

$$N^*_{\gamma+1}(k,s)\left[S^{\alpha} + i\delta^{1/k} C_{\gamma+1} k\right] = S^{\beta(\alpha-1)} f^*_{\gamma+1}(k) + \Phi^*_{\gamma+1}(k,s)$$

$$N^*_{\gamma+1}(k,s) = \frac{S^{\beta(\alpha-1)} f^*_{\gamma+1}(k)}{\left[S^{\alpha}+i\delta^{1/k} C_{\gamma+1} k\right]} + \frac{\Phi^*_{\gamma+1}(k,s)}{\left[S^{\alpha}+i\delta^{1/k} C_{\gamma+1} k\right]} \quad (26)$$

Taking the inverse Laplace transform of the both side

$$N^*_{\gamma+1}(k,t) = f^*_{\gamma+1}(k) t^{\alpha-\beta(\alpha-1)-1} {}_pM_q^{\alpha,\alpha-\beta(\alpha-1)}\left(-iC_{\gamma+1} k t^{\alpha}\right) + \int_0^t \Phi^*_{\gamma+1}(k, t-\xi) \xi^{\alpha-1} {}_pM_q^{\alpha,\alpha}\left(-iC_{\gamma+1} k \xi^{\alpha}\right) d\xi \quad (27)$$

From Chaurasia and Singh [3]

$$L^{-1}\left\{\frac{S^{\alpha-1}}{[S^{\beta}+a]}\right\} = t^{\beta-\alpha} E_{\beta,\beta-\alpha+1}(-at^{\beta}) \quad (28)$$

In the similar way

$$L^{-1}\left\{\frac{S^{\beta(\alpha-1)+1-1}}{\left(\left[S^{\alpha}+i\delta^{1/k} C_{\gamma+1} k\right]\right)}\right\} = t^{\alpha-\beta(\alpha-1)-1} {}_pM_q^{\alpha,\alpha-\beta(\alpha-1)}\left(-iC_{\gamma+1} k t^{\alpha}\right) \quad (29)$$

Taking the inverse Fourier transform and using fractional Fourier transform, Convolution theorem

$$N(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-ik(x-\tau)} \left[t^{\alpha-\beta(\alpha-1)-1} {}_pM_q^{\alpha,\alpha-\beta(\alpha-1)}\left(-iC_{\gamma+1} k t^{\alpha}\right) f(\tau) d\tau\right] dk$$
$$+ \frac{1}{2\pi} \int_0^t \int_{-\infty}^{\infty} \left\{\int_{-\infty}^{\infty} e^{-ik(x-\tau)} [(t-\xi)^{\alpha-1} {}_pM_q^{\alpha,\alpha}\left(-iC_{\gamma+1} k (t-\xi)^{\alpha}\right) d\xi] dk\right\} \Phi(\tau,\xi) d\tau$$

$$N(x,t) = \int_{-\infty}^{\infty} G_1(x-\tau,t) f(\tau) d\tau + \int_0^t (t-\xi)^{\alpha-1} \int_{-\infty}^{\infty} G_2(x-\tau, t-\xi) \Phi(\tau,\xi) d\tau$$

Where

$$G_1(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ikx} t^{\alpha-\beta(\alpha-1)-1} {}_pM_q^{\alpha,\alpha-\beta(\alpha-1)}\left(-iC_{\gamma+1} k t^{\alpha}\right) dk$$

$$G_2(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ikx} {}_pM_q^{\alpha,\alpha}\left(-iC_{\gamma+1} k t^{\alpha}\right) dk$$

5. Special Cases:-

Corollary 5.1:- When we put $\Phi(x,t) = te^{-a|x|}$, $a > 0$ in the theorem (4.1)

$${}_0D_t^{\alpha,\beta} N(x,t) = \delta^{1/k} D_{\eta}^{\gamma+1} N(x,t) + te^{-a|x|}$$

$$, x \in R, t > 0, \delta > 0, k > 0 \quad (30)$$

Under the initial condition

$$I_{0+}^{(1-\beta)(1-\alpha)} N(x, 0+) = f(x) \quad (31)$$

Then we get

$$N(x,t) = \int_{-\infty}^{\infty} G(x-\tau,t) f(\tau) d\tau + \frac{1}{\pi}\left\{\int_0^t (t-\xi)^{\alpha-1} d\xi \left[\int_{-\infty}^{\infty} e^{-ikx}\left\{\left[(t-\xi) - \frac{sina\,(t-\xi)}{a}\right] {}_pM_q^{\alpha,\alpha}\left(-iC_{\gamma+1} k (t-\xi)^{\alpha}\right)\right\} dk\right]\right\} \ldots (32)$$

Where

$$G(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ikx} t^{\alpha-\beta(\alpha-1)-1} {}_pM_q^{\alpha,\alpha-\beta(\alpha-1)}\left(-iC_{\gamma+1} k t^{\alpha}\right) dk$$

Corollary 5.2:- When we put $\delta^{1/k} = \mu$ in the theorem (4.1). We derived the something similar result of Chaurasia and Singh [3]

Corollary 5.3:- When we put $\beta = 0$ in the theorem (4.1)

$${}_0D_t^{\alpha} N(x,t) = \delta^{1/k} D_{\eta}^{\gamma+1} N(x,t) + \Phi(x,t) \quad , x \in R, t > 0, \delta > 0, k > 0 \quad (33)$$

Under the initial condition

$${}_0D_t^{\alpha-1} N(x,0) = f(x) \quad (34)$$

Then we get

$$N(x,t) = \int_{-\infty}^{\infty} G_1(x-\tau,t) f(\tau) d\tau + \int_0^t (t-\xi)^{\alpha-1} d\xi \int_{-\infty}^{\infty} G_2(x-\tau, t-\xi) \Phi(\tau,\xi) d\tau \quad (35)$$

Where $G_1(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ikx} t^{\alpha-1} {}_pM_q^{\alpha,\alpha}\left(-iC_{\gamma+1} k t^{\alpha}\right) dk$

$$G_2(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ikx} {}_pM_q^{\alpha,\alpha}\left(-iC_{\gamma+1} k t^{\alpha}\right) dk$$

Corollary 5.4:- When we put $\beta = 0$ and $\alpha = \frac{1}{2}$ in the theorem (4.1). We have derived the following similar result of the Mainardi, Luchko and Pagnini [7]

$$ {}_0D_t^{1/2} N(x,t) = \delta^{1/k} D_\eta^{\gamma+1} N(x,t) + \Phi(x,t) \quad , x \in R, t > 0, \delta > 0, k > 0 \qquad (36)$$

Under the initial condition

$$N(x,0) = f(x) \qquad (37)$$

Then we get

$$N(x,t) = \int_{-\infty}^{\infty} G_1(x-\tau,t) f(\tau) d\tau + \int_0^t (t-\xi)^{\alpha-1} d\xi \int_{-\infty}^{\infty} G_2(x-\tau, t-\xi) \Phi(\tau,\xi) d\tau \ \ldots (38)$$

Where $G_1(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ikx} t^{\alpha-\frac{1}{2}} {}_pM_q^{\alpha,\alpha+\frac{1}{2}}\left(-iC_{\gamma+1} k t^{\alpha}\right) dk$

$$G_2(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ikx} \, {}_pM_q^{\alpha,\alpha}\left(-iC_{\gamma+1} k t^{\alpha}\right) dk$$

Corollary 5.5:-If we put $\beta = 1$ in the theorem (4.1). Then we have derived the following similar result is given by Nikolova and Boyadjiev [9]

The Cauchy type problem for the fractional diffusion equation

$$ {}_0^cD_t^{\alpha} N(x,t) = \delta^{1/k} D_\eta^{\gamma+1} N(x,t) + \Phi(x,t), x \in R, t > 0, 0 < \alpha < 1, 0 < \gamma < 1, \delta > 0, k > 0 \qquad (39)$$

Under the initial condition $N(x,0) = f(x)$ (40)

Then we get

$$N(x,t) = \int_{-\infty}^{\infty} G_1(x-\tau,t) f(\tau) d\tau + \int_0^t (t-\xi)^{\alpha-1} \int_{-\infty}^{\infty} G_2(x-\tau, t-\xi) \Phi(\tau,\xi) d\tau \qquad (41)$$

Where $G_1(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ikx} \, {}_pM_q^{\alpha,1}\left(-iC_{\gamma+1} k t^{\alpha}\right) dk$

$$G_2(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ikx} \, {}_pM_q^{\alpha,\alpha}\left(-iC_{\gamma+1} k t^{\alpha}\right) dk$$

6. Conclusion:

In this paper, we have studied the Cauchy type problem for the fractional diffusion equation with non-linear function. We have found the fundamental solution of the Cauchy type problem are also expressed in the form of theorem in terms of M-series.

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