ISSN: 2456-1657

Elins International Journal

Elins International Journal of Science Engineering & Management (EIJSEM), Volume 4, Issue 5, June 2020

INEQUALITIES RELATED TO MATRIX NORMS

Dragan Obradovic[1], Lakshmi Narayan Mishra[2*]

[1]Elementary school "Jovan Cvijic", Kostolac-Pozarevac,
teacher of Mathematics, Serbia dragishaobradovic@yahoo.com

[2]Department of Mathematics, School of Advanced Sciences,VIT University, Vellore 632 014, Tamil Nadu, India
lakshminarayanmishra04@gmail.com

Abstract: The matrix of inequalities represents a very diverse, but also extremely extensive area mathematics. Therefore, in this paper we will focus on observing inequalities with degrees, eigenvalues and singular values, as well as matrix norms. Regardless of the debt and a rich history of the development of this field of mathematics, matrix inequalities are still very much a current topic dealt with by many prominent mathematicians of today.

1. INTRODUCTION

In this paper, an overview of the presented statements is given and possible generalizations are pointed out obtained results. Further work can be based on generalizations of proven inequalities for certain types of operators, as well as on the observation of other inequalities that apply to the set complex numbers.

Definition. *Let X be a complex vector space.*
The function $\|\cdot\|: X \to \mathbb{R}$ *is the norm in space X if the following conditions are met:*

(i) $\|x\| \geq 0$ *for everyone* $x \in X$;

(ii) $\|x\| = 0$ *if and only if it is* $x = 0$;

(iii) $\|\lambda x\| = |\lambda|\|x\|$ *for everyone* $x \in X$ *i svako* $\lambda \in \mathbb{C}$;

(iv) $\|x + y\| \leq \|x\| + \|y\|$ *all* $x, y \in X$.

Different norms are used in matrix spaces. First of all, how do we look at matrices and as operators, the operator norm should be noted. We will draw attention to others later examples of norms.

In this chapter, we will present some of the more important inequalities related to matrix norms. This article aims to consider inequalities involving uniquely invariant norms C. He, L. Zou [1]. Emphasis is placed on generalizations of known inequalities that apply to a set of real, that is, complex numbers. We will point out different approaches to generalizations.

Above all, we will show the generalizations of some basic inequalities that apply to the set complex numbers. On a set of complex numbers, the inequality is satisfied for every
$x \in \mathbb{R}$ $|z - \text{Re } z| \leq |z - x|$, $z \in \mathbb{C}$. The corresponding generalization to the matrix space is given in the following statement. Each matrix represents a linear mapping O. Milinković, Z. Petrić [2].

Proposition *Inequality*

$$\|A - \text{Re } A\| \leq \|A - H\|$$

applies to each Hermitian matrix H and $A \in M_n$.

Proof.

$$\|A - \text{Re}A\| = \|A - \frac{1}{2}(A + A^*)\| = \frac{1}{2}\|(A - H) + (H - A^*)\|$$

$$\leq \frac{1}{2}\|A - H\| + \frac{1}{2}\|H - A^*\| = \frac{1}{2}\|A - H\| + \frac{1}{2}\|(H - A)^*\|$$

$$= \|A - H\| .$$

The inequality $\|\text{R}_e A\| \leq \|A\|$, $A \in$ Mn is an obvious generalization of the inequality $|\text{Re } z| \leq |z|$ for $z \in \mathbb{C}$.

Proposition If the matrices $A, B \in M_n$ are positive de nits, then:

$$\|A^r + B^r\| \leq \|(A + B)^r\|, r \geq 1;$$

$$\|A^r + B^r\| \geq \|(A + B)^r\|, 0 \leq r \leq 1.$$

Proof. Let us fix m ≥ 0. Let Ωm be the set of real numbers in the interval [1, *m*] for which inequality met The set Ω_m is closed, and not empty because $1 \in \Omega m$. We will show that it is related to it. More precisely, we illustrate its use in geometric separation, N. Teofanov [3].

For r, s ∈ Ω_m let $t = \frac{r+s}{2}$. From

$$\begin{bmatrix} A^t + B^t & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} A^{r/2} & B^{r/2} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} A^{s/2} & 0 \\ B^{s/2} & 0 \end{bmatrix}$$

completely valid

$$\|A^t + B^t\| \le \| \begin{bmatrix} A^{r/2} & B^{r/2} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} A^{s/2} & 0 \\ B^{s/2} & 0 \end{bmatrix} \|.$$

Based on the properties of the norm $\|X\| = \|X^* X\|^{1/2} = \|XX^*\|^{1/2}$ we have

[1] Here it is considered that the reader is familiar with the basic concepts of the topological structure of a set of real numbers.

$$\begin{aligned} \|A^t + B^t\| &\le \|A^r + B^r\|^{1/2} \|A^2 + B^s\|^{1/2} \\ &\le \|(A+B)^r\|^{1/2} \|(A+B)^s\|^{1/2} \\ &= \|A+B\|^{r/2} \|A+B\|^{s/2} \\ &= \|A+B\|^t = \|(A+B)^t\|. \end{aligned}$$

Therefore $t \in \Omega_m \Rightarrow \Omega_m = [1, m]$, which we have shown.

The second inequality is obtained by applying precisely.

For $0 < r \le 1$, $\frac{1}{r} \ge 1$, so it is

$$\|(A^r)^{1/r} + (B^r)^{1/r}\| \le \|(A^r + B^r)^{1/r}\| = \|A^r + B^r\|^{1/r}.$$

which is equivalent to $\| (A+B)^r\| \le \|A^r + B^r\|$.

2. UNITARY INVARIANT NORMS

The norm ‖ · ‖ in the spaceM_n is unitarily invariant if $\|UAV\| = \|A\|$ for all A, U, $V \in M_n$ where U and B are unitary matrices. The perturbation result for the singular values enlarges the class of well-behaved matrices for accurate computation of the singular values V. Hari [4].

We will point out the consequence of the polar decomposition matrix theorem. Let $A = SW$ polar decomposition of matrix A, W unitary matrix. The matrix S is positively semidefinite, and there exists a unitary matrix Y such that $S = Y * \Lambda Y$, Λ is a diagonal matrix. Combining these two equations we get $A = Y * \Lambda Y W$. The matrices $Y *$ and YW are unitary.

Thus, for the matrix $A \in M_n$ there are unitary matrices U and V, as well as a diagonal matrix Λ such that $A = U\Lambda V$.

Based on the Spectral Decomposition Theorem for each matrix $A \in M_n$ there are unitary matrices U, V for which $A = U \operatorname{diag}(s_1(A),..., s_n(A)) V$. Hence they are unitary invariant norms of the singular value function. Decomposition into singular values has a number of important algebraic and geometric properties G. V. Milovanovic, R. Ž. Djordjevic [5].

Definition. Mapping $\Phi: \mathbb{R}^n \to \mathbb{R}$ is a symmetric gage function if they are satisfied the following conditions:

(i) $\Phi(P_x) = \Phi(x)$, *for all permutation matrices P and each* $x \in \mathbb{R}^n$;

(ii) $\Phi(\varepsilon_1 x_1,..., \varepsilon_n \tau^n) = \Phi(x_1,..., \tau_n)$ for $\varepsilon_i = \pm 1$, $i = 1, n$ and $x = (x_1,..., \tau_n) \in \mathbb{R}^n$.

John von Neumann[1] showed that unitary invariant norms can be identified with symmetric gage functions[2].

Thus, there is a "1−1" mapping from the set of unitarily invariant norms $\| \cdot \|_\Phi$ and symmetric gage function Φ. The observed functional dependence can contain a large number of logically redundant functional dependencies M. Blagojević [6].

$$\|A\|_\Phi = \Phi(s_1(A), \ldots, s_n(A)) \; A \in M_n.$$

A significant example of unitary invariant norms are the Fan [3] k-norms, $\| \cdot \|_k$.

$$(\forall A \in M_n) \; \|A\|_{(k)} = \sum_{i=1}^{k} s_j(A), \; 1 \le k \le n.$$

In accordance with the introduced notation is $\| \cdot \|_{(1)} = \| \cdot \|_\infty$ i $\| \cdot \|_{(n)} = \| \cdot \|_1$.

Definition. Let ‖ · ‖ *be a norm in the space* M_n. *Dual norm of norms* $\| \cdot \|$, $\| \cdot \|^D$,
de nisana is with

$$\|A\|^D := \max\{|tr(AB^*)| : \|B\| = 1, B \in M_n\}.$$

The dual norm is also the norm on M_n. The dual norm of a unitary invariant norm is unitary invariant. Equivalent definition of a dual norm that generalizes the notion of intensity of a geometric vector D. S. Đorđević [7].

$$\|A\|D := \max\{ \sum_{i=1}^{k} s_j(A)s_j(B) : \|B\| = 1, B \in M_n\}.$$

That's right $(\| \cdot \|^D)^D = \| \cdot \|$.

Definitely for $A \in M_n$ norm $\| \cdot \|_\gamma$.

ISSN: 2456-1657

$$\|A\|_{\gamma} := \sum_{i=1}^{k} \gamma_j s_j(A)$$

Wherein $\gamma = (\gamma_1, \ldots, \gamma_n)$, $\gamma_1 \geq \cdots \geq \gamma_n$ i $\gamma_1 > 0$. The norm defined in this way is unitary invariant.

Lemma. Let $\| \cdot \|$ be a unitarily invariant norm in space Mn i $\Gamma = \{s(X) : \|X\|^D = 1, X \in M_n\}$. Then for all $A \in M_n$

$$\|A\| = \max\{\|A\| : \gamma \in \Gamma\}$$

Lema. (Fan's principle of dominance) Let $A, B \in M_n$. From the conditions

$$\|A\|_{(k)} \leq \|B\|_{(k)}, k = 1, 2, \ldots, n$$

Follows

$$\|A\| \leq \|B\|$$

for any unitary invariant norm.

Proof. Evidence follows on the basis of Lemma and identity

$$\|A\|_{\gamma} = \sum_{i=1}^{k} (\gamma_j - \gamma_{j+1})\|A\|_{(j)} + \gamma_n\|A\|_{(n)}.$$

3. CONCLUSION

As we have already pointed out, matrix inequalities are a very broad and diverse area of development has been going on for centuries. In this paper, the processing inequalities related to degrees are singular and eigenvalues of matrices, as well as matrix norms, with an emphasis on unitarily invariant norms.

Also, further work could be related to the observation of inverse inequalities, as well as to the generalizations of other inequalities that are satisfied on a set of real or complex numbers.

REFERENCES

[1] C. He, L. Zou, Some inequalities involving uniquely invariant norms, Mathematical inequalitiesand Applications, 4 (2012), 767-77672

[2] O. Milinković and Z. Petrić, Linear Algebra Script January 2013. Department of Informatics, Faculty of Mathematics, University of Belgrade.

[3] N. Teofanov, Elementary Mathematics 2 (M-08) March 27, 2018.

[4] V. Hari, MATRIX THEORY OF PERTURBATION, UNIVERSITY OF ZAGREB, Zagreb, 1996.

[5] G. V. Milovanović; R. Ž. Djordjevic LINEAR ALGEBRA Nis, September 30, 2004.

[6] M. Blagojević, GRAVITY AND LOCAL SYMMETRY, INSTITUTE OF PHYSICS - BELGRADE, 1997.

[7] D. S. Đorđević, Finally dimensional vector spaces, Niš, 2012.